The Double Angle Formulae

In general, we have found when working with functions that $f(x_1 + x_2)$ is not necessarily equal to $f(x_1) + f(x_2)$. It is true for some functions (eg $f(x) = kx$) but not others (eg $f(x) = x^2$). Is it true for trig functions? Let's try some numbers and see.

Eg $\quad f(x) = \cos(x)$

$$x_1 = 0°$$

$$x_2 = 90°$$

We have $\cos(0° + 90°) = \cos(90°) = 0$

but $\cos(0°) + \cos(90°) = 1 + 0 = 1 \neq 0$ hence

$$\cos(0° + 90°) \neq \cos(0°) + \cos(90°)$$

We just need one counterexample to disprove something, so now we know that in general we cannot expect $\cos(x_1 + x_2)$ to be equal to $\cos(x_1) + \cos(x_2)$.

We can do the same thing for sine:

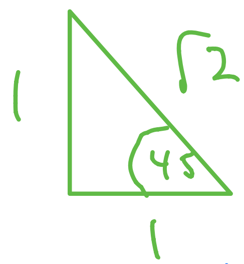
Eg $\quad f(x) = \sin(x)$

$x_1 = 45$

$x_2 = 45$

We have $\sin(45° + 45°) = \sin(90°) = 1$
but $\sin(45°) + \sin(45°) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$.
Hence

$$\sin(45°) + \sin(45°) = \sqrt{2} \neq 1 = \sin(45° + 45°)$$

[Note 

$\sin(45) = \frac{1}{\sqrt{2}}$]

Finally for tan:

Eg $f(x) = \tan(x)$

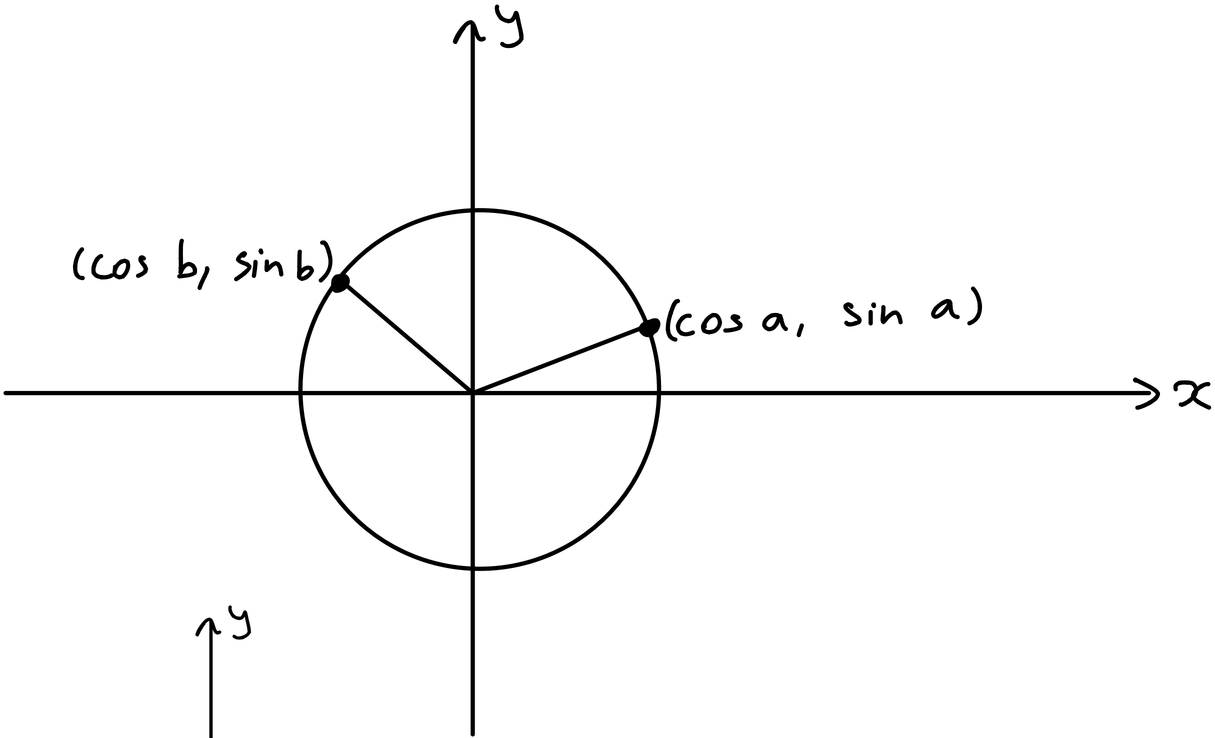
$x_1 = 45°$

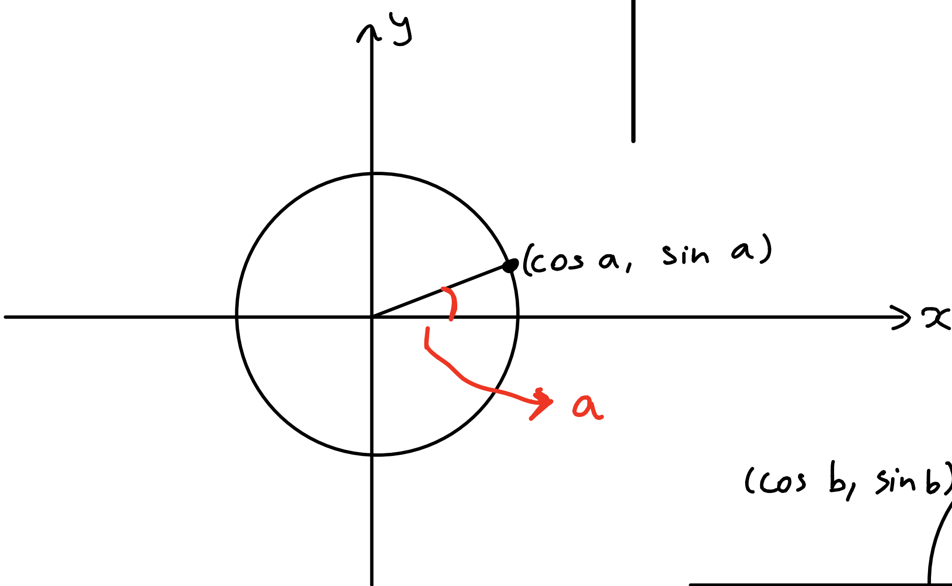
$x_2 = 45°$

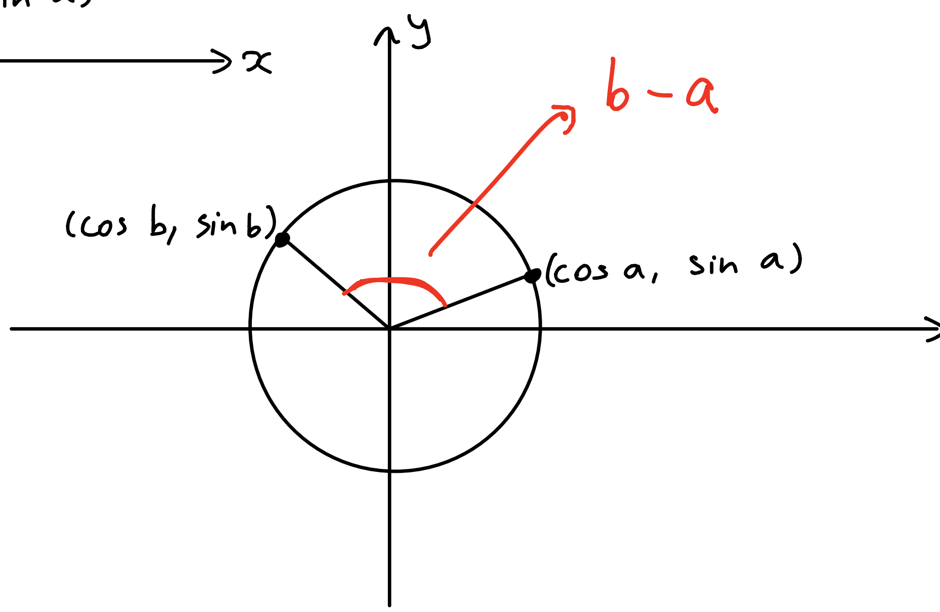
We have $\tan(45° + 45°) = \tan(90°)$ which is undefined! But $\tan(45°) + \tan(45°) = 1 + 1 = 2$.

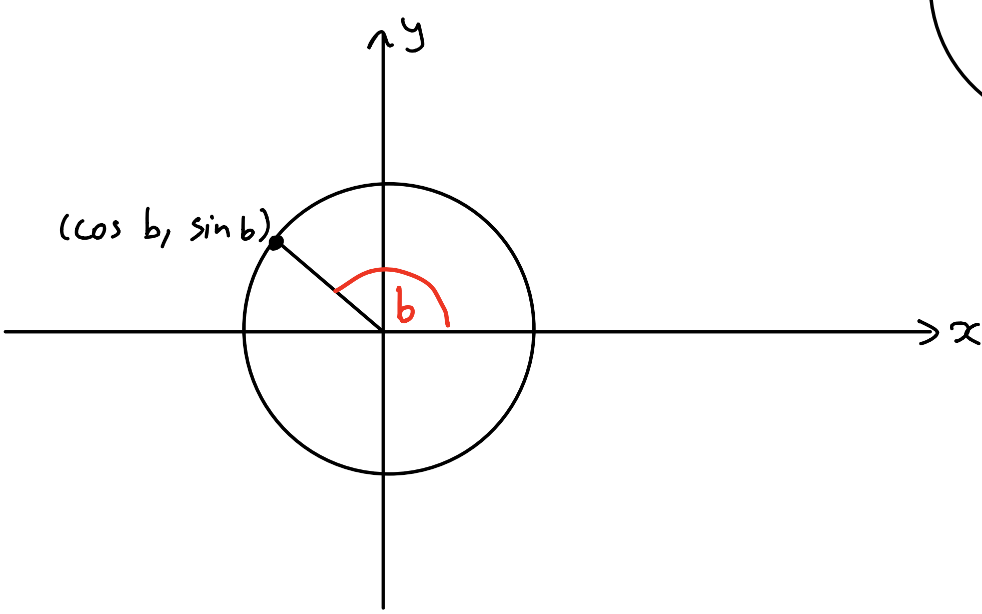
So the next question is, can we find a convenient formula for $f(a \pm b)$ where f is a trig function?

Let's start out with the coordinate axes and unit circle approach to trig:



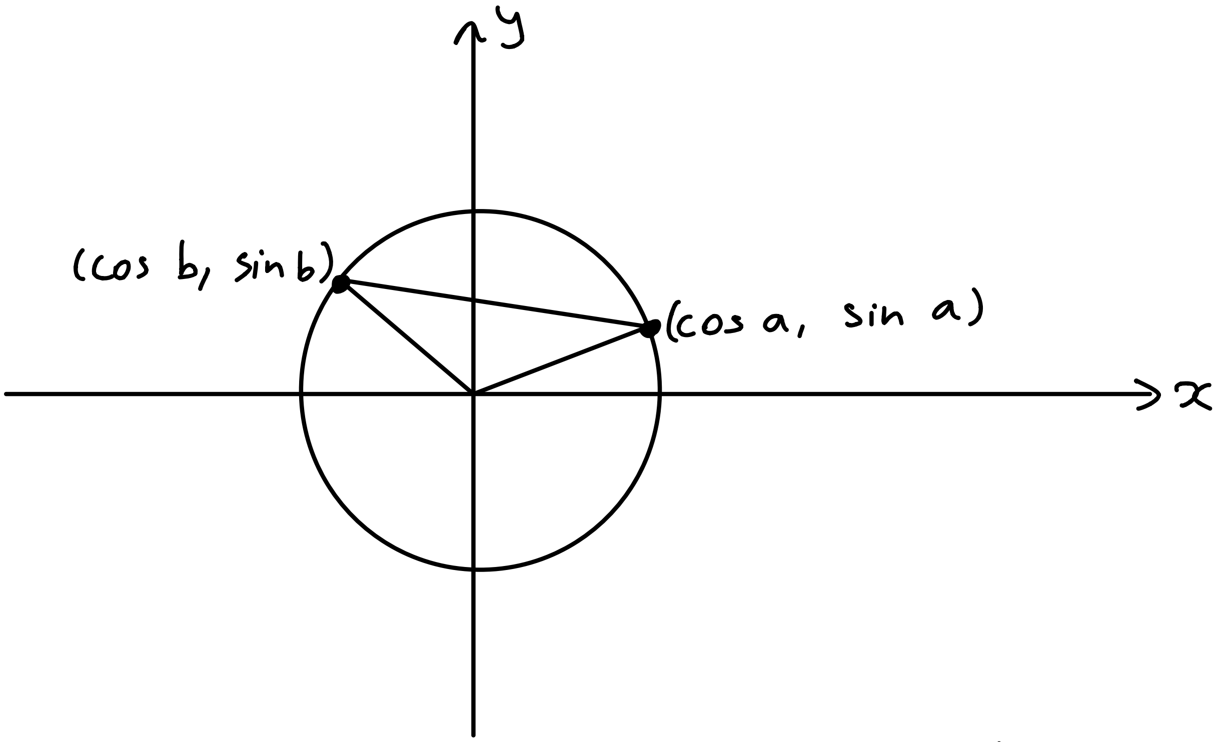






So we have obtained an angle of $b-a$ at the centre of the circle.

Now, we connect the two points on the circle in order to create a triangle:



Using the Distance Formula, we have

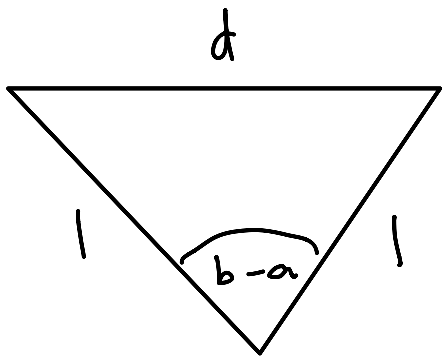
$$d^2 = (\sin(b) - \sin(a))^2 + (\cos(b) - \cos(a))^2$$

$$= \sin^2(b) + \sin^2(a) - 2\sin(a)\sin(b) + \cos^2(b) + \cos^2(a) - 2\cos(a)\cos(b)$$

$$= [\sin^2(b) + \cos^2(b)] + [\sin^2(a) + \cos^2(a)] - 2[\sin(a)\sin(b) + \cos(a)\cos(b)]$$

$$= 2 - 2[\sin(a)\sin(b) + \cos(a)\cos(b)]$$

Now let's look at the triangle from a Euclidean geometry perspective:



We know that the two sides have length 1 since they are radii of the unit circle. (Look at the previous diagram).

We now apply the Cosine Rule :

$$d^2 = 1^2 + 1^2 - 2(1)(1)\cos(b-a)$$
$$= 2 - 2\cos(b-a)$$

Finally, we equate our two expressions for d^2 :

$$2 - 2[\sin(a)\sin(b) + \cos(a)\cos(b)] = 2 - 2\cos(b-a)$$

Rearrange :

$$\boxed{\cos(b-a) = \sin(a)\sin(b) + \cos(a)\cos(b)}$$

Now what about $\cos(b+a)$? We don't have to do all that work again, we can simply sub $(-a)$ where we previously had $(+)a$:

$$\cos(b+a) = \cos(b-(-a))$$
$$= \sin(-a)\sin(b) + \cos(-a)\cos(b)$$

$$= -\sin(a)\sin(b) + \cos(a)\cos(b)$$

[Recall that sin is an odd function while cos is an even function]

So

$$\boxed{\cos(b+a) = \cos(a)\cos(b) - \sin(a)\sin(b)}$$

Now what about $\sin(b \pm a)$?

Again we can use our previous results. We have

$$\cos(b) = \cos(b+(a-a))$$

$$= \cos([b-a]+a)$$

$$= \cos(b-a)\cos(a) - \sin(b-a)\sin(a)$$

Rearrange to get $\sin(b-a)$ by itself:

$$\sin(b-a)\sin(a) = \cos(b-a)\cos(a) - \cos(b)$$

$$\Rightarrow \sin(b-a) = \frac{\cos(b-a)\cos(a) - \cos(b)}{\sin(a)}$$

Expand $\cos(b-a)$ using above results:

$$\sin(b-a) = \frac{[\cos(a)\cos(b) + \sin(a)\sin(b)]\cos(a) - \cos(b)}{\sin(a)}$$

$$= \frac{\cos^2(a)\cos(b)}{\sin(a)} + \sin(b)\cos(a) - \frac{\cos(b)}{\sin(a)}$$

$$= \frac{\cos(b)}{\sin(a)}\left[\cos^2(a) - 1\right] + \sin(b)\cos(a)$$

$$= \frac{\cos(b)}{\sin(a)}\left[-(1-\cos^2(a))\right] + \sin(b)\cos(a)$$

$$= \frac{\cos(b)}{\sin(a)}\left[-\sin^2(a)\right] + \sin(b)\cos(a)$$

$$= -\cos(b)\sin(a) + \sin(b)\cos(a)$$

Hence we have the result

$$\boxed{\sin(b-a) = \sin(b)\cos(a) - \cos(b)\sin(a)}$$

For $\sin(b+a)$ we can use the same trick as for cos above:

We have

$$\sin(b+a) = \sin(b-(-a))$$

$$= \sin(b)\cos(-a) - \cos(b)\sin(-a)$$

$$= \sin(b)\cos(a) + \cos(b)\sin(a)$$

[Again recalling that cos is even and sin is odd]

So

$$\boxed{\sin(b+a) = \sin(b)\cos(a) + \cos(b)\sin(a)}$$

Now for tan we don't have to do much extra work, simply recall that $\tan(x) = \frac{\sin(x)}{\cos(x)}$:

So $\tan(b-a) = \dfrac{\sin(b-a)}{\cos(b-a)}$

$$= \frac{\sin(b)\cos(a) - \cos(b)\sin(a)}{\cos(b)\cos(a) + \sin(b)\sin(a)}$$

Now we divide top and bottom by $\cos(a)\cos(b)$ [exactly the same as multiplying by 1]:

So

$$\tan(b-a) = \frac{\dfrac{\sin(b)\cos(a)}{\cos(a)\cos(b)} - \dfrac{\cos(b)\sin(a)}{\cos(a)\cos(b)}}{\dfrac{\cos(b)\cos(a)}{\cos(a)\cos(b)} + \dfrac{\sin(b)\sin(a)}{\cos(a)\cos(b)}}$$

$$= \frac{\dfrac{\sin(b)}{\cos(b)} - \dfrac{\sin(a)}{\cos(a)}}{1 + \left(\dfrac{\sin(a)}{\cos(a)}\right)\left(\dfrac{\sin(b)}{\cos(b)}\right)}$$

$$\therefore \boxed{\tan(b-a) = \frac{\tan(b) - \tan(a)}{1 + \tan(a)\tan(b)}}$$

Again we use the $a \leftrightarrow -a$ technique to find an expression for $\tan(b+a)$:

We have

$$\tan(b+a) = \tan(b-(-a))$$

$$= \frac{\tan(b) - \tan(-a)}{1 + \tan(-a)\tan(b)}$$

$$\therefore \boxed{\tan(b+a) = \frac{\tan(b) + \tan(a)}{1 - \tan(a)\tan(b)}}$$

Now, when $a = b$ we have some special results called the Double Angle Formulae:

$$\begin{aligned}
\cos(2a) &= \cos(a)\cos(a) - \sin(a)\sin(a) \\
&= \cos^2(a) - \sin^2(a) \\
&= \cos^2(a) - [1 - \cos^2(a)] \\
&= 2\cos^2(a) - 1 \\
&= [1 - \sin^2(a)] - \sin^2(a) \\
&= 1 - 2\sin^2(a)
\end{aligned}$$

make everything into sin

make everything into cos

$$\begin{aligned}
\sin(2a) &= \sin(a)\cos(a) + \cos(a)\sin(a) \\
&= 2\sin(a)\cos(a)
\end{aligned}$$

$$\begin{aligned}
\tan(2a) &= \frac{\tan(a) + \tan(a)}{1 - \tan(a)\tan(a)} \\
&= \frac{2\tan(a)}{1 - \tan^2(a)}
\end{aligned}$$

As you can see, the Double Angle Formula for cos has many different forms... in any given situation you can select the one which is most useful to you.